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Technical Notes

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Analytical Solution for Hyperbolic Heat Conduction in a Hollow Sphere

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Nomenclature

a	=	thermal diffusivity, $m^2\ s^{-1}$
c	=	velocity of the thermal propagation, $m\ s^{-1}$
F_1, F_2	=	intermediate functions
f_1, f_2	=	source functions of F_1, F_2
$H(\,)$	=	Heaviside's unit step function
$I_n(\,)$	=	modified Bessel function of the first kind and order n
$L(\,)$	=	Laplace transform
$L^{-1}(\,)$	=	inverse Laplace transform
l	=	mean free path of a molecule, m
p	=	dimensionless quantity to designate position of the wave front
q	=	heat flux, $W\ m^{-2}$
r	=	radial or spatial coordinate, m
r_i	=	inner radius, m
r_o	=	outer radius, m
r_γ	=	relative thickness of the hollow sphere, $= r_i/r_o$
S	=	heat source, $W\ m^{-3}$
s	=	Laplace transform variable
T	=	temperature, K
T_{wi}	=	temperature of inner surface, K
T_{wo}	=	temperature of outer surface, K
T_γ	=	relative temperature change, $= (T_{wi} - T_0)/(T_{wo} - T_0)$
T_0	=	initial temperature, K
t	=	time, s
v	=	velocity of phonon or electron, $m\ s^{-1}$
β	=	intermediate function
ε	=	dimensionless characteristic time, $= a\tau/r_o^2$
η	=	dimensionless position $(= r/r_o)$
θ	=	dimensionless temperature, $= (T - T_0)/(T_{wo} - T_0)$
λ	=	thermal conductivity, $W\ m^{-1}\ K^{-1}$
ξ	=	dimensionless time, $= at/r_o^2$
τ	=	thermal characteristic (or relaxation) time, s

Superscript

$\sim$	=	Laplace transformed function

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Introduction

THE well-known Fourier's heat-conduction law assumes that the temperature difference ∇T and heat propagation $\boldsymbol{q}$ take place concurrently, thus implying an infinite speed of propagation for a thermal disturbance, indicating that a local change of temperature causes an instantaneous perturbation at each point in the medium. This is physically unreasonable. Accounting for the lagging response in time between the heat-flux vector and the temperature gradient, Tzou[1] provided a macroscopic formulation to describe the nonequilibrium thermodynamic transition, which can be expressed by

$$\boldsymbol{q}(\boldsymbol{r}, t+\tau) = -\lambda \nabla T(\boldsymbol{r}, t) \tag{1}$$

Equation (1) indicates the temperature gradient established at time t yields a heat-flux vector at a later time $t+\tau$ because of a time of response delay. By applying a Taylor-series expansion and by neglecting the second- and higher-order derivatives, a hyperbolic energy equation for thermal processes in micro- and nanoscale and/or in an ultrafast timescale reads as

$$\nabla^2 T(\boldsymbol{r}, t) + \frac{1}{\lambda}\left[S(\boldsymbol{r}, t) + \tau\frac{\partial S(\boldsymbol{r}, t)}{\partial t}\right] = \frac{1}{a}\left[\frac{\partial T(\boldsymbol{r}, t)}{\partial t} + \tau\frac{\partial^2 T(\boldsymbol{r}, t)}{\partial t^2}\right] \tag{2}$$

This is the so-called hyperbolic non-Fourier heat-conduction model,[2] from which the thermal propagation is deduced to have a wave nature.

Much effort has been spent to obtain solutions of the hyperbolic heat-conduction equation under different conditions and to develop mathematical and numerical techniques that can accurately predict the non-Fourier temperature profiles for a wide range of physical, geometric, and boundary conditions.[3–10] The pursuit of analytical solutions for the hyperbolic heat-conduction equations is of intrinsic scientific interest. In the present Note, an analytical expression for the temperature profile in a hollow sphere with sudden temperature changes on its inner and outer surface is derived. Based on this expression, the hyperbolic heat propagation behavior in the hollow sphere is analyzed, and several hyperbolic heat-conduction anomalies are identified and discussed in detail.

Model Description

A radial one-dimensional heat-conduction process is considered for a hollow sphere with an inner radius r_i and outer radius r_o and with constant thermal properties and uniform initial temperature distribution T_0. The thermal disturbance is caused by sudden temperature changes on its inner surface (from T_0 to T_{wi}) and outer surface (from T_0 to T_{wo}); no heat source is involved. The hyperbolic heat-conduction equation, Eq. (2), thus takes the following dimensionless form:

$$\frac{\partial^2\theta(\eta, \xi)}{\partial\eta^2} + \frac{2}{\eta}\frac{\partial\theta(\eta, \xi)}{\partial\eta} = \frac{\partial\theta(\eta, \xi)}{\partial\xi} + \varepsilon\frac{\partial^2\theta(\eta, \xi)}{\partial\xi^2} \tag{3}$$

with initial conditions

$$\theta(\eta, 0) = 0, \quad \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=0} = 0 (r_\gamma \le \eta \le 1) \tag{4}$$

and boundary conditions

$$\theta(r_\gamma, \xi) = T_\gamma, \theta(1, \xi) = 1 (\xi > 0) \tag{5}$$

where

$$\theta = (T - T_0)/(T_{\mathrm{wo}} - T_0), \qquad \eta = r/r_o, \qquad \xi = at\big/r_o^2$$

$$\varepsilon = a\tau\big/r_o^2, \qquad r_\gamma = r_i/r_o, \qquad \text{and} \qquad T_\gamma = (T_{\mathrm{wi}} - T_0)/(T_{\mathrm{wo}} - T_0)$$

The thermal relaxation time τ can be expressed as $\tau = 3a/v^2 = a/c^2$ because the velocity v of phonon or electron is equal to l/τ; the dimensionless parameter ε then can be defined as $\varepsilon = l^2/3r_o^2$, that is, the ratio of thermal length scale to the characteristic physical length scale.

Analytical Solution

Applying the Laplace transform to Eq. (3) with respect to the variable ξ and taking into account the initial conditions expressed by Eq. (5), a subsidiary equation yields

$$\frac{\partial^2\tilde{\theta}(\eta, s)}{\partial\eta^2} + \frac{2}{\eta}\frac{\partial\tilde{\theta}(\eta, s)}{\partial\eta} - (s + \varepsilon s^2)\tilde{\theta}(\eta, s) = 0 \tag{6}$$

with boundary conditions

$$\tilde{\theta}(r_\gamma, s) = T_\gamma/s, \qquad \tilde{\theta}(1, s) = 1/s \tag{7}$$

The solution of Eq. (6) with the conditions of Eq. (7) is

$$\begin{aligned}\eta\tilde{\theta}(\eta, s) &= \frac{1}{s}\left\{\frac{\exp\left[\sqrt{s+\varepsilon s^2}(\eta-1)\right] - T_\gamma r_\gamma \exp\left[\sqrt{s+\varepsilon s^2}(\eta+r_\gamma-2)\right]}{1-\exp\left[-2\sqrt{s+\varepsilon s^2}(1-r_\gamma)\right]}\right\} \\ &+ \frac{1}{s}\left\{\frac{T_\gamma r_\gamma \exp\left[\sqrt{s+\varepsilon s^2}(r_\gamma-\eta)\right] - \exp\left[\sqrt{s+\varepsilon s^2}(2r_\gamma-\eta-1)\right]}{1-\exp\left[-2\sqrt{s+\varepsilon s^2}(1-r_\gamma)\right]}\right\}\end{aligned} \tag{8}$$

Replacing the denominators inside the braces of Eq. (8) with a series of $\sqrt{(s+\varepsilon s^2)}$ as

$$\frac{1}{1-\exp\left[-2\sqrt{s+\varepsilon s^2}(1-r_\gamma)\right]} = \sum_{n=0}^{\infty}\exp\left[-2n\sqrt{s+\varepsilon s^2}(1-r_\gamma)\right] \tag{9}$$

it yields,

$$\eta\tilde{\theta}(\eta, s) = \frac{1}{s}\sum_{n=0}^{\infty}\begin{pmatrix}\exp\left[-\sqrt{s+\varepsilon s^2}(2n-2nr_\gamma-\eta+1)\right] \\ -T_\gamma r_\gamma \exp\left\{-\sqrt{s+\varepsilon s^2}[2n+2-\eta-(2n+1)r_\gamma]\right\} \\ +T_\gamma r_\gamma \exp\left\{-\sqrt{s+\varepsilon s^2}[2n+\eta-(2n+1)r_\gamma]\right\} \\ -\exp\left\{-\sqrt{s+\varepsilon s^2}[2n+\eta+1-2(n+1)r_\gamma]\right\}\end{pmatrix} \tag{10}$$

The inverse Laplace transform gives for the temperature response in the hollow sphere the following expression:

$$\eta\theta(\eta, \xi) = L^{-1}[\eta\tilde{\theta}(\eta, s)] = L^{-1}[s\tilde{\beta}(\eta, s)] = \frac{\partial\beta(\eta, \xi)}{\partial\xi} \tag{11}$$

The source function $\beta(\eta, \xi)$ is determined from

$$\beta(\eta, \xi) = L^{-1}[\tilde{\beta}(\eta, s)] = L^{-1}[F_1(\eta, s)F_2(s)] = f_1(\eta, \xi)^* f_2(\xi) \tag{12}$$

with

$$F_1(\eta, s) = \frac{1}{\sqrt{s+\varepsilon s^2}} \times \sum_{n=0}^{\infty}\begin{pmatrix}\exp\left[-\sqrt{s+\varepsilon s^2}(2n-2nr_\gamma-\eta+1)\right] \\ -T_\gamma r_\gamma \exp\left\{-\sqrt{s+\varepsilon s^2}[2n+2-\eta-(2n+1)r_\gamma]\right\} \\ +T_\gamma r_\gamma \exp\left\{-\sqrt{s+\varepsilon s^2}[2n+\eta-(2n+1)r_\gamma]\right\} \\ -\exp\left\{-\sqrt{s+\varepsilon s^2}[2n+\eta+1-2(n+1)r_\gamma]\right\}\end{pmatrix} \tag{13}$$

$$F_2(s) = \frac{\sqrt{s+\varepsilon s^2}}{s^2} \tag{14}$$

The convolution integral is defined as

$$f_1(\eta, \xi)^* f_2(\xi) = \int_0^{\xi} f_1(\eta, \xi') f_2(\xi - \xi')\,\mathrm{d}\xi' \tag{15}$$

Because the source functions $f_1(\eta, \xi)$ and $f_2(\xi)$ can be determined from a table of inverse Laplace transforms, finally, the expression for the hyperbolic temperature propagation in the hollow sphere yields

$$\begin{aligned}\eta\theta(\eta, \xi) &= e^{-\xi/2\varepsilon}\sum_{n=0}^{\infty}\begin{pmatrix} I_0\left[\frac{1}{2\varepsilon}\sqrt{\xi^2-(2n-2nr_\gamma-\eta+1)^2\varepsilon}\right]H\left[\xi-\sqrt{\varepsilon}(2n-2nr_\gamma-\eta+1)\right] \\ -T_\gamma r_\gamma I_0\left\{\frac{1}{2\varepsilon}\sqrt{\xi^2-[2n-(2n+1)r_\gamma-\eta+2]^2\varepsilon}\right\}H\left\{\xi-\sqrt{\varepsilon}[2n-(2n+1)r_\gamma-\eta+2]\right\} \\ +T_\gamma r_\gamma I_0\left\{\frac{1}{2\varepsilon}\sqrt{\xi^2-[2n-(2n+1)r_\gamma+\eta]^2\varepsilon}\right\}H\left\{\xi-\sqrt{\varepsilon}[2n-(2n+1)r_\gamma+\eta]\right\} \\ -I_0\left\{\frac{1}{2\varepsilon}\sqrt{\xi^2-[2n-2(n+1)r_\gamma+\eta+1]^2\varepsilon}\right\}H\left\{\xi-\sqrt{\varepsilon}[2n-(2n+1)r_\gamma+\eta+1]\right\}\end{pmatrix} + \frac{1}{2\varepsilon}e^{-\xi/2\varepsilon} \\ &\times \sum_{n=0}^{\infty}\int_0^{\xi}\begin{pmatrix} I_0\left[\frac{1}{2\varepsilon}\sqrt{\xi'^2-(2n-2nr_\gamma-\eta+1)^2\varepsilon}\right]H\left[\xi'-\sqrt{\varepsilon}(2n-2nr_\gamma-\eta+1)\right] \\ -T_\gamma r_\gamma I_0\left\{\frac{1}{2\varepsilon}\sqrt{\xi'^2-[2n-(2n+1)r_\gamma-\eta+2]^2\varepsilon}\right\}H\left\{\xi'-\sqrt{\varepsilon}[2n-(2n+1)r_\gamma-\eta+2]\right\} \\ +T_\gamma r_\gamma I_0\left\{\frac{1}{2\varepsilon}\sqrt{\xi'^2-[2n-(2n+1)r_\gamma+\eta]^2\varepsilon}\right\}H\left\{\xi'-\sqrt{\varepsilon}[2n-(2n+1)r_\gamma+\eta]\right\} \\ -I_0\left\{\frac{1}{2\varepsilon}\sqrt{\xi'^2-[2n-2(n+1)r_\gamma+\eta+1]^2\varepsilon}\right\}H\left\{\xi'-\sqrt{\varepsilon}[2n-(2n+1)r_\gamma+\eta+1]\right\}\end{pmatrix} \\ &\times \left[I_0\left(\frac{\xi-\xi'}{2\varepsilon}\right) + I_1\left(\frac{\xi-\xi'}{2\varepsilon}\right)\right]\mathrm{d}\xi'\end{aligned} \tag{16}$$

Taking the limiting situation, $r_\gamma \to 0$, that is, $r_i \to 0$ or $r_o \to \infty$ to test the reliability of the preceding expression, it goes back to the form of the temperature solution for a solid sphere.[6]

Hyperbolic Heat Propagation Behaviour

A hollow sphere of $r_\gamma = 0.6$ is considered; the same temperature change amount is prescribed at its two free surfaces, that is, $T_\gamma = 1$. On the basis of Eq. (16), numerical calculations were performed to display the thermal performance of the hollow sphere.

In what concerns three materials of $\varepsilon = 0.05$, 0.15, and 0.25, respectively, temperature variations at the central plane inside the hollow sphere ($\eta = 0.8$) are depicted in Fig. 1. For the material of $\varepsilon = 0.05$, the temperature in the hollow sphere gradually tends toward the final thermal equilibrium value (1.0), and no evident wave characteristic is detected, which is an indication that the classic Fourier thermal diffusion theory is still valid. In contrast, the temperature variations for the materials of $\varepsilon = 0.15$ and 0.25 take well-defined wave characteristics. The temperature wave rapidly attenuates as time advances. Approximately after $\xi > 3.0$, no temperature wave can be detected, and the leading temperature values gradually approach the final thermal equilibrium value (1.0). Thereafter, the Fourier heat-conduction law is again valid.

The temperature distribution in the hollow sphere, as depicted in Fig. 2, clearly displays the time-delay phenomenon, which characterizes the hyperbolic heat propagation. According to Eq. (16), the time-delay amount is directly proportional to $\varepsilon^{1/2}$. Therefore, a material of relatively larger ε value, namely, $\varepsilon = 0.5$, is considered with the purpose of enhancing the hyperbolic thermal responses. With the advance of time, the two thermal waves, respectively originating from the two free surfaces transmit to and from in the hollow sphere yielding complicated processes, such as reflection by the boundary surfaces and wave interference, take place.

Several hyperbolic heat-conduction anomalies are specially depicted in Figs. 1 and 2. "A1" indicates an unacceptably high (>1.0) temperature region that can appear in the central part inside the hollow sphere; "A2" points to a region where temperature is increasing, whereas the temperature gradient is zero. In the region circled by "A3," which is tightly close to the heated outer boundary surface, often the temperature gradient is negative, whereas the heat is transported inside from this boundary; "A4" highlights a region close to the inner boundary surface, where the temperature gradient is, after $\xi > 0.6$, zero or slightly positive, whereas the heat moves inward. Similar phenomena have been reported in the literature[6,8–10] for hyperbolic heat-conduction processes in finite bodies imposed by various thermal disturbance conditions, but they were often ignored and explained as natural results caused by hyperbolic heat propagation. Reasonable explanations were given by Haji-Sheikh et al.[5] In their work, these phenomena are deemed as heat-conduction anomalies, and they appear when a wave reaches a boundary, or

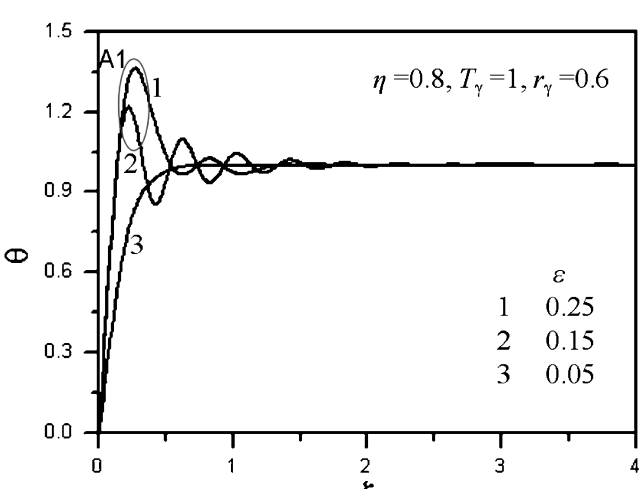

Fig. 1 Hyperbolic temperature waves.

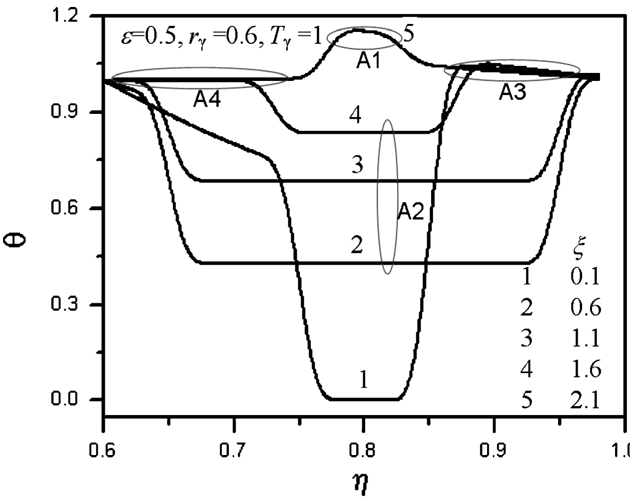

Fig. 2 Temperature distribution in the hollow sphere.

when two waves interact in a domain, or when other discontinuities exist.

With closer observation to the thermal process shown in Fig. 2, the position where the heat wave front arrives is determined by a dimensionless quantity p:

$$p = ct/(r_o - r_i) = \xi \big/ (1 - r_\gamma)\sqrt{\varepsilon} \tag{17}$$

Thus, $\xi = 0.1$, 0.6, 1.1, 1.6, and 2.1 correspond to $p = 0.35$, 2.12, 3.89, 5.66, and 7.42, respectively. For $\xi = 0.1$, the two heat waves do not yet encounter; there is a thermal static region existing in the central part of the hollow sphere. When $\xi = 0.6$, the heat waves already travel a round trip in the hollow sphere; they hit the inner and outer boundary surface each one time and meet each other at the central plane inside the hollow sphere two times. Every time the wave hits the boundary wall, an energy pulse is produced, and the Dirichlet boundary condition can be violated.[5] It is the violation of the boundary conditions that results in anomalies A3 and A4. The interaction between the two waves also results in an energy pulse,[5] which forces the occurrence of anomaly A2. Anomalies A2, A3, and A4 are accentuated when the heat waves hit the boundary surfaces and meet each other several times. Anomaly A1 is, essentially, an extremely enhanced A2. Special attention is paid to the anomaly A3. It forms as early as $\xi = 0.1$. When $\xi = 0$, the Dirichlet boundary condition is imposed, and a heat wave (or energy pulse) is initiated from the outer boundary. Because of the decreasing surface area along the wave propagation path, the energy pulse intensity is thus augmented, which is not properly accommodated by the temperature solution, Eq. (16). This reveals the anomaly A3 is set in at the beginning of this thermal process.

Haji-Sheikh et al.[5] proposed a general method to deal with an anomaly by replacing the temperature discontinuity with an equivalent volumetric heat source for inclusion in the temperature solution. Cheng[7] derived a new type of equations and named them ballistic-diffusive equations, which has the potential of giving more reasonable non-Fourier temperature representations relative to the hyperbolic heat-conduction equation. The procedure required to handle the hyperbolic heat conduction anomalies is lengthy and is not included in this particular work.

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